Static and Dynamic Analysis of a Massless Scalar Field Coupled with a Class of Gravity Theories

Youngjai Kiem

Joseph Henry Laboratories

Princeton University

Princeton, NJ 08544

And

Dahl Park

Department of Physics

KAIST

Taejon 305-701, KOREA

E-mail: dpark@chep5.kaist.ac.kr
ykiem@puhep1.princeton.edu

Abstract

General static solutions for a massless scalar field coupled to a class of effectively 2-d gravity theories continuously connecting spherically symmetric d-dimensional Einstein gravity $(d > 3)$ and the CGHS model are analytically obtained. They include black holes and point scalar charge solutions with naked singularities, and are used to give an analytic proof of no-hair theorem. Exact scattering solutions in s-wave 4-d Einstein gravity are constructed as a generalization of corresponding static solutions. They show the existence of black hole formation threshold for square pulse type incoming stress-energy flux, above which trapped surfaces are dynamically formed. The relationship between this behavior and the numerically studied phase transition in this system [1] is discussed.

1. Introduction

Recently, there has been remarkable progress in black hole physics, as we gain valuable insights from model gravity theories such as the CGHS model where the theory is exactly integrable [2]. In these simplified settings, we can address such vexing questions as the presumed information loss via Hawking radiation process, black hole thermodynamics and related issues in a much more concrete analytical way. Even some developments on quantization were manageable [3]. At the same time, progress has also been made in 4-d Einstein gravity coupled with various matter fields. Especially interesting approach, to name an example, is the study of the gravitational collapse in spherically symmetric reduction of 4-d Einstein gravity coupled with a single massless scalar field [4]. Several rigorous mathematical results and some explicit solutions were obtained so far [9] [13]. Moreover, an intriguing critical behavior was observed in this system near the onset of the black hole formation through numerical analysis [1]. A generic 1-parameter class of solutions $S[p]$ decomposes into two phases, depending on the magnitude of p that measures the strength of self-gravitational interaction. For $p < p^*$, where p^* is the critical value, the gravitational collapse of incoming stress-energy flux is followed by an explosion, reflecting back the flux toward the future infinity. If $p > p^*$, the black holes are dynamically formed and the mass of the asymptotically static black hole M_{BH} shows a universal scaling behavior, $M_{BH} \simeq |p - p^*|^\Delta$ with $\Delta \simeq 0.37$. The analytic explanation for this phase transition has been only partially successful [12] [5], although in the context of the (1-loop corrected) CGHS model the analytic explanation for the similar phase transition with $\Delta = 0.5$ is available [6].

As has been well established now, the spherically symmetric 4-d Einstein gravity and the CGHS model, both of them (effectively) 2-dimensional, can be given a unified treatment as a generalized 2-d dilaton gravity theory. To be specific, we can construct a class of 2-d gravity theories that continuously connect these two interesting theories [11]. In view of this point, our interest in analyzing these interconnecting theories is at least two-folds; first, we would like to know to exactly what extent many interesting results obtained for the CGHS model are the generic lessons on gravitational physics and what results are artifacts of the particular choice of parameters. Secondly, since the explicit analytic calculations in 4-d Einstein gravity are very difficult, we want to find a simpler setting to answer many classical (and possibly quantum) questions in general relativity in detail. This can possibly teach us more about classical and quantum cosmological questions such as the realistic

gravitational collapsing process in general relativity.

Keeping these considerations in mind, in this paper, we systematically study classical general static solutions for a scalar field coupled with various effectively 2-d gravity theories that continuously connect spherically symmetric 4-d Einstein gravity and the CGHS model. The model action we consider here is given as follows

$$I = \int d^2x \sqrt{-g} e^{-2\phi} (R^{(2)} + \gamma g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \mu e^{2\lambda\phi} - \frac{1}{2} e^{-2\phi(\delta-1)} g^{\alpha\beta} \partial_\alpha f \partial_\beta f) \qquad (1)$$

and we focus on the case of $\delta = 1$ for analytic tractability. Here $R^{(2)}$ denotes the 2-d scalar curvature. ϕ and f represent a dilaton field and a massless scalar field, respectively. γ, λ, δ and μ are real parameters. The specific choice of these 4 parameters corresponds to a particular gravity theory. Originally, the action introduced above was considered as a 2-d target space effective action resulting from string theory, if we neglect loop corrections. Other interpretations of the action were found since then. After $(d-2)$-dimensional angular integration, for example, the action for spherically symmetric d-dimensional Einstein gravity theories reduces to Eq.(1), modulo total derivative terms, with $\gamma = 4(d-3)/(d-2)$, $\lambda = 2/(d-2)$ and $\delta = 1$. The dilaton field ϕ is related to the geometric radius r of the transversal $(d-2)$-sphere as $r = \exp(-2\phi/(d-2))$. On the other hand, the CGHS model is recovered if we set $\gamma = 4$, $\lambda = 0$ and $\delta = 0$. Thus, we see that as long as the pure gravity sector is concerned, the CGHS model is exactly the same as $d \to \infty$ limit of d-dimensional Einstein gravity. The only difference is the dilaton prefactor in the action for the scalar field in the latter theory.

In section 2, we derive the general static solutions for our model with $\delta = 1$ using a rather elementary method. Specifically, for general value of δ, we integrate the equations of motion, a system of highly non-linear second-order coupled differential equations, by finding the relevant number of symmetries of the action, reducing the order of differential equations by one. Thus, the result of integration can be represented as the conservation of several Noether charges. These equations are more analytically tractable and, for $\delta = 1$ and $2 - \lambda - \gamma/4 > 0$, i.e. $d > 3$, they can be further integrated exactly to yield closed form expressions. The main novelty here is our method of derivation that enables us to get the *general* static solutions. Unlike a well-developed technique of generating solutions for complicated gravity theories from vacuum solutions of Einstein equations, our method solves equations of motion directly, thereby getting general solutions in a straightforward and simple fashion. Our analysis shows that the space of all possible static solutions modulo coordinate

transformation is two dimensional (or several sheets of two dimensional space). The space is parameterized, roughly, by black hole mass and scalar charge. For non-vanishing black hole mass, in 4-d Einstein gravity, we reproduce the exact solutions obtained by Janis *et.al.*[7] long time ago using a solution generating technique from the vacuum solutions of Einstein equations. Recalling that solutions we get here are general static solutions, our further analysis leads to a new analytic proof of no-hair theorem that takes the backreaction on space-time geometry non-perturbatively into account. More interesting result is the case of vanishing black hole mass. In this case, there is a general relativistic analog of a (non-relativistic) point scalar charge solution in the vacuum space-time in each theory. Particularly, these solutions contain naked singularities. Although they are solutions for which the total gravity-matter action in d-dimensional Einstein gravity

$$I^{(d)} = \int d^d x \sqrt{-g^{(d)}} (R^{(d)} - \frac{1}{2} g^{(d)\alpha\beta} \partial_\alpha f \partial_\beta f)$$

vanishes, it remains to be seen whether they are stable against classical and quantum perturbations.

In section 3, we find the extension of the static point scalar charge solutions into scattering solutions is easy in some cases, namely, the spherically symmetric reduction of 4-d Einstein gravity. Utilizing the extension, we construct dynamic solutions, which demonstrate that, for the square pulse type incoming stress-energy flux, there exists a black hole formation threshold for the incoming stress-energy flux T_{in}. Specifically, in our case, the threshold value of the incoming stress-energy flux, T_{in}^c, that plays the role of p^* is found to be

$$\lim_{r \to \infty} \int_{angle} T_{in}^c / c^2 = 4\pi T_{in}^c / c^2 = \frac{c^3}{4G},$$

where G is the gravitational constant and c is the speed of light. If $T_{in} > T_{in}^c$, black holes are dynamically formed (supercritical case, in the language of phase transition). Then, an injection of shock wave type stress-energy flux naturally belongs to the supercritical case, for the value of T_{in} is very large during a very short period of time, and the corresponding solutions are constructed. Below the threshold, $T_{in} < T_{in}^c$, the incoming pulse type stress-energy flux gets reflected forward into the future null infinity (subcritical case). In supercritical case, we compute the critical exponent for our solutions taking the apparent mass of the resulting black hole immediately after the formation as an order parameter, getting $M_A \simeq |p - p^*|^{0.5}$. We further indicate that our analysis here supports the aforementioned numerical study [1] by showing

4

a plausible mechanism that generates a non-trivial scaling relation between M_A and M_{BH}. The difficulties of the similar extension in other generic gravity theories are then discussed. Based on the qualitative arguments, we later conjecture that the scaling relation for d-dimensional Einstein gravity for $d > 4$ is $M_{BH} \simeq |p - p^*|^{0.5}$, unlike the 4-dimensional (and 3-dimensional [8]) results. This dependence of the critical exponent on the space-time dimensionality is very similar to the well-known cases in condensed matter physics, such as the ferromagnetic transition.

In section 4, we discuss our results and their possible physical implications in various physical contexts.

2. General Static Solutions

We start by deriving general static solutions in conformal gauge. The solutions we get in this section can also be obtained under a different gauge choice as sketched in Appendix. The calculations in Appendix show that, in case of spherically symmetric 4-d Einstein gravity coupled with a massless scalar field, the solutions for non-vanishing black hole mass are identical to those found in [7] using a solution generating technique.

2.1. Derivation of General Static Solutions

The equations of motion we have to solve to get the static solutions are obtained from our action by varying it with respect to the metric tensor, dilaton field and the massless scalar field;

$$D_\alpha D_\beta \Omega - g_{\alpha\beta} D \cdot D\Omega + \frac{\gamma}{8}(g_{\alpha\beta}\frac{(D\Omega)^2}{\Omega} - 2\frac{D_\alpha \Omega D_\beta \Omega}{\Omega}) + \frac{\mu}{2}g_{\alpha\beta}\Omega^{1-\lambda} \qquad (2)$$

$$+\frac{1}{2}\Omega^\delta D_\alpha f D_\beta f - \frac{1}{4}\Omega^\delta g_{\alpha\beta}(Df)^2 = 0$$

$$R + \frac{\gamma}{4}(\frac{(D\Omega)^2}{\Omega^2} - 2\frac{D \cdot D\Omega}{\Omega}) + (1-\lambda)\mu\Omega^{-\lambda} - \frac{\delta}{2}\Omega^{\delta-1}(Df)^2 = 0 \qquad (3)$$

$$g^{\alpha\beta}D_\alpha(\Omega^\delta D_\beta f) = 0, \qquad (4)$$

where we define $\Omega = e^{-2\phi}$ and D denotes the covariant derivative. Since the only matter coupling in our case is a massless scalar field, we choose to work in a conformal gauge. Thus, we write the metric $g_{\alpha\beta} = -\exp(2\rho + \frac{\gamma}{2}\phi)dx^+dx^-$. Our convention, for the calculational simplicity, is the negative signature for space-like coordinates and the positive signature for a time-like coordinate. In spherically symmetric 4-d Einstein gravity, for example, this implies $\mu = -2$ using Gauss-Bonnet theorem. Under this choice of coordinates, the original action gets simplified to become

$$I = \int dx^+ dx^- (4\Omega\partial_+\partial_-\rho + \frac{\mu}{2}e^{2\rho}\Omega^{1-\lambda-\gamma/4} + \Omega^\delta\partial_+ f\partial_- f), \tag{5}$$

modulo total derivative terms. In our choice of the conformal factor, we included a contribution from the dilaton field. This contribution, up to a total derivative term we threw away, was so chosen to cancel the kinetic energy term for the dilaton field, rendering a simplified form of the action. Eq.(5) should be supplemented by gauge constraints resulting from the choice of the conformal gauge. They are calculated to be

$$\partial_\pm^2\Omega - 2\partial_\pm\rho\partial_\pm\Omega + \frac{1}{2}\Omega^\delta(\partial_\pm f)^2 = 0, \tag{6}$$

from Eq.(2) for $g_{\pm\pm}$ components of the metric tensor.

The general static solutions for the equations of motion from the action (5) under the gauge constraints can be found as follows; we can consistently reduce the partial differential equations (PDE's) into the coupled second order ordinary differential equations (ODE's) by assuming all functions depends on a single space-like coordinate $x = x^+x^-$ *. Although it is not, in general, possible to have a consistent reduction of PDE's to ODE's assuming an arbitrary one-variable dependence, our choice turns out to be consistent. This procedure yields the following ODE's

$$x\ddot{\Omega} + \dot{\Omega} + \frac{\mu}{4}e^{2\rho}\Omega^{1-\lambda-\gamma/4}\dot{\rho} = 0, \tag{7}$$

$$x\ddot{\rho} + \dot{\rho} + \frac{\mu}{8}(1 - \lambda - \gamma/4)\frac{e^{2\rho}}{\Omega^{\lambda+\gamma/4}} + \frac{\delta}{4}x\Omega^{\delta-1}\dot{f}^2 = 0,$$

$$\frac{d}{dx}(x\Omega^\delta\dot{f}) = 0,$$

along with the gauge constraint

$$\ddot{\Omega} - 2\dot{\rho}\dot{\Omega} + \frac{1}{2}\Omega^\delta\dot{f}^2 = 0,$$

---

*This coordinate becomes space-like due the signature choice made here

where the dot represents taking a derivative with respect to x. The complete general solutions of the above ODE's are the same as the general static Gravity-Scalar solutions under a particular choice of the conformal coordinates. The solutions under a different choice of conformal coordinates can be obtained from the solutions of Eqs.(7) by proper conformal transformations of $x^{\pm}$. If we set $f = 0$, i.e., if we consider the pure gravity sector, Birkhoff's theorem ensures that the solutions obtained in this fashion are indeed general solutions of the original PDE's. The equations of motion other than the gauge constraint can be summarized by an action

$$I = \int dx (x\dot{\Omega}\dot{\rho} - \frac{\mu}{8}e^{2\rho}\Omega^{1-\lambda-\gamma/4} - \frac{1}{4}x\Omega^{\delta}\dot{f}^2). \tag{8}$$

One can straightforwardly verify that by varying this action with respect to Ω, ρ and f, we recover Eqs.(7).

We observe that the action (8) has two obvious rigid continuous symmetries. First symmetry is clear, for f field appears only through its first derivative. Thus, we see that $f \to f + \alpha$ is a symmetry for an arbitrary constant α. The second symmetry is the transformation $x \to xe^{\alpha}$ and $\rho \to \rho - \alpha/2$. This symmetry represents the scaling invariance of the action (8). The existence of this symmetry is necessary since the original action (5) is invariant under the (local) conformal transformations. Since there are three functions ρ, Ω and f that we should solve in terms of x, we can integrate Eqs.(7) once to reduce them to first order ODE's if we can find one additional rigid continuous symmetry of the action. The remaining symmetry turns out to be $x \to x^{1+\alpha}$, $\rho \to \rho - (2 - \lambda - \gamma/4)\ln(1+\alpha)/2 - \alpha \ln x/2$, $\Omega \to \Omega(1+\alpha)$, and $f \to f(1+\alpha)^{(1-\delta)/2}$. This transformation changes the action (8) by a total derivative. We can deduce the form of this symmetry by the following physical consideration; the asymptotically flat spatial coordinate at spatial infinity is related to the spatial coordinate that is flat near the black hole horizon by the conformal transformation $x^{\pm} \to \ln x^{\pm}$, as is familiar from the definition of the tortoise coordinate. The physical system in our consideration is also scale invariant in this asymptotically flat region. Our third symmetry is this logarithmic conformal coordinate transformation followed by rescaling of Ω and f fields, representing this asymptotic scale invariance.

Given these three symmetries, we can construct the corresponding Noether charges.

$$f_0 = x\Omega^{\delta}\dot{f} \tag{9}$$

$$c_0 = x^2 \dot{\rho} \dot{\Omega} + \frac{1}{2} x \dot{\Omega} - \frac{1}{4} x^2 \Omega^\delta \dot{f}^2 + \frac{\mu}{8} x e^{2\rho} \Omega^{1-\lambda-\gamma/4} \tag{10}$$

$$M + 2c_0 \ln x = 2x \dot{\rho} \dot{\Omega} - (2 - \lambda - \gamma/4) x \dot{\Omega} + \Omega + \frac{\delta - 1}{2} x \Omega^\delta f \dot{f} \tag{11}$$

We can rewrite Eqs.(7) in a form that represents the conservation of Noether charges f_0, c_0 and M. By integrating this form of equations of motion, we get the equations shown above where f_0, c_0 and M are constants of integration. Additionally, the gauge constraint reduces to a condition $c_0 = 0$. In the absence of matter fields, modulo coordinate transformations, the general solutions of gravity theories considered here are parameterized by a single parameter, i.e., the black hole mass. Similarly, in our case the gauge constraint kills a redundant degree of freedom, c_0, for gravity sector. As will be clear in the following sections, two remaining Noether charges f_0 and M can be related to a scalar charge and a black hole mass, respectively.

We can further solve the above equations and this process can be straightforwardly carried out for $\delta = 1$. In this case, we can solve ρ from Eq.(11) and f from Eq.(9) to find

$$\rho = \frac{M}{2} \int \frac{dy}{\Omega} + \frac{1}{2}(1 + q) \ln \Omega - \frac{1}{2} y + \rho_0 \tag{12}$$

and

$$f = f_0 \int \frac{dy}{\Omega} + f_1, \tag{13}$$

where we define $y = \ln x$ and $q = 1 - \lambda - \gamma/4$. Here ρ_0 and f_1 are additional constants of integration. For spherically symmetric reduction of d-dimensional Einstein gravity, we have $1 + q = (d-3)/(d-2)$. Plugging Eqs.(12) and (13) into Eq.(10), we get a decoupled equation for Ω, an integro-differential equation.

$$M \frac{d\Omega}{dy} + (1 + q)(\frac{d\Omega}{dy})^2 - \frac{1}{2} f_0^2 = -\frac{\mu}{4} \Omega^{2+2q} \exp(M \int \frac{dy}{\Omega} + 2\rho_0) \tag{14}$$

Our goal is to solve this equation to get as explicitly as possible the closed form expression of Ω in terms of y. By differentiating it with respect to y we have

$$\frac{d^2}{dy^2} \Omega = (M \frac{d}{dy} \Omega + (1 + q)(\frac{d}{dy} \Omega)^2 - \frac{1}{2} f_0^2) \frac{1}{\Omega}, \tag{15}$$

a second order ODE. Integrating Eq.(15) once is immediate; we find

$$k\Omega^{1+q} = |h_+ - \frac{d\Omega}{dy}|^{\frac{h_+}{h_+ + h_-}} |h_- + \frac{d\Omega}{dy}|^{\frac{h_-}{h_+ + h_-}} \tag{16}$$

$$\equiv F_{h_-, h_+}(\frac{d\Omega}{dy})$$

where we define a function F_{h_-,h_+} and two numbers

$$h_\pm = \frac{1}{2(1+q)}(\sqrt{M^2 + 2(1+q)f_0^2} \mp M) \geq 0$$

are introduced. We note that k is a constant of integration introduced when we go from Eq.(15) to Eq.(16). Under the assumption $1 + q > 0$, the asymptotic analysis of Eq.(16) shows that the resulting space-time is asymptotically flat. In this situation, a rather complicated intro-differential equation (14) can be replaced by much simpler Eq.(16). Further asymptotic analysis shows that the equivalence is insured if we identify

$$k = (-\frac{\mu e^{2\rho_0}}{4(1+q)})^{1/2}$$

If we properly choose the range of Ω and $\frac{d\Omega}{dy}$, we can define the inverse of the function F_{h_-,h_+}. Then, the integration of Eq.(16) is trivially performed to yield

$$\ln(x/x_0) = \int d\Omega \frac{1}{F_{h_-,h_+}^{-1}(k\Omega^{1+q})} \tag{17}$$

where x_0 is a constant of integration.

Eqs.(12), (13) and (17) are general static solutions of our problem. Under our gauge choice, our solutions are parameterized by 5 parameters, M, f_0, ρ_0, f_1 and x_0, since the gauge constraint mandates $c_0 = 0$. Among these, f_1 is just an addition of a constant term to f, which is trivial. The constants of integration ρ_0 and x_0 represent the degree of freedom in the choice of coordinate system, namely, the global scale choice and the reference time choice, respectively. Thus, modulo coordinate transformations and the trivial f_1 part, we find the general static solutions of Gravity-Scalar action in our consideration are parameterized by two parameters f_0, the scalar charge, and M, which will be shown to be related to the black hole mass.

2.2. No-hair Theorem and Point Scalar Charges

In our further consideration of general static solutions, we focus on the cases when $M \geq 0$ and $f_0 \geq 0$. The restriction on M is motivated by the fact that we want the black hole mass to be positive semi-definite.[†] The results for negative values of f_0 can be trivially obtained, as is clear from Eqs.(12), (13) and (16).

[†]In fact, all the solution with $M < 0$ contains unphysical naked singularities as one can convince oneself from the general static solutions.

The derivation in the previous section shows that getting the inverse function of F_{h_-,h_+} is necessary to determine the relationship between Ω and $y = \ln x$ (note that we have $x = x^+ x^-$ in our choice of conformal coordinates). Under our restrictions, there are four distinctive cases we have to consider as shown in Figs.1a - 1d for the shape of the function F_{h_-,h_+}. In each figure, the physical region to take the inverse of F_{h_-,h_+} is shown; since the value of F_{h_-,h_+} relates to some power of geometric radius of transversal sphere, via Eq.(16), we require it to vary from zero to infinity. Furthermore, we require $\frac{d\Omega}{dy}$ to be positive for large values of Ω, thereby, F_{h_-,h_+}. In other words, $F_{h_-,h_+} = 0$ is a natural space-time boundary because it represents a space-time point where the transversal sphere collapses to a point. Similarly the value $F_{h_-,h_+} = \infty$ corresponds to asymptotic spatial infinities. In every figure, F_{h_-,h_+} grows linearly for large, positive $\frac{d\Omega}{dy}$, showing the spatial infinity regions are asymptotically flat. Thus, our choice of physical region is tantamount to considering only static solutions with asymptotically flat space-time region at spatial infinities and with the conventional choice for the orientation of mappings between Ω and conformal coordinates there. By making this choice of physical region, we are no longer considering parts of general static solutions that have compact range of geometric radius. The existence of them is clear from Figs.1c and 1d. These solutions with no asymptotic infinities can be potentially important in some physical settings such as cosmological considerations. In this note, however, since we will eventually be interested in constructing scattering solutions with asymptotic in/out-regions, we restrict our attention to space-time geometries with flat asymptotic infinities.
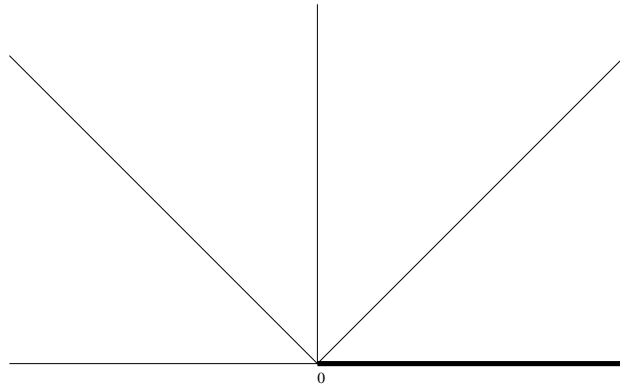

Fig.1a. The plot of F_{h_-,h_+} as a function of $\frac{d\Omega}{dy}$ for $h_+ = h_- = 0$. The region to take the inverse of F_{h_-,h_+} is depicted by a bold line.
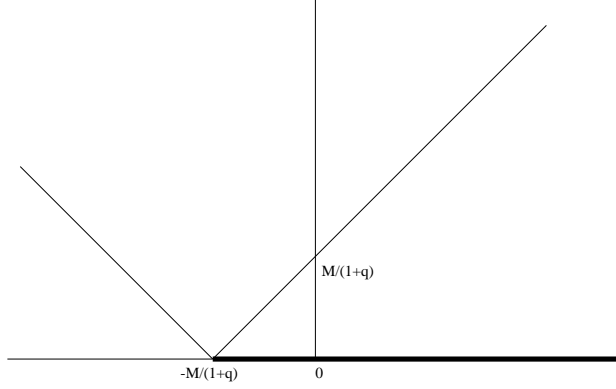
M/(1+q)

-M/(1+q)   0

Fig.1b. The plot of F_{h_-,h_+} as a function of $\frac{d\Omega}{dy}$ for $h_+ = 0$ and $h_- = M/(1+q)$, a black hole geometry with mass M. The region to take the inverse of F_{h_-,h_+}, a bold line, contains the point $\frac{d\Omega}{dy} = 0$.

In Fig.1a, we have $M = 0$ and $f_0 = 0$, implying $h_\pm = 0$. Then the scalar field f vanishes identically and this is the case that corresponds to the vacuum solution in each theory. For non-vanishing value of q, we have

$$\Omega = (-qk(y - y_0))^{-1/q}. \tag{18}$$

For d-dimensional Einstein gravity, where we have $q = -1/(d-2)$, this solution represents the expression for the geometric radius in term of conformal coordinates. For example, in 4-d case, the above expression becomes $r^2 = ((\ln x^+ + \ln x^-)/2)^2$ while the examination of Eq.(12) shows that $\ln x^\pm$ are asymptotically flat conformal coordinates. For $q = 0$, we find

$$\ln \Omega = k(y - y_0), \tag{19}$$

a familiar linear dilaton vacuum in pure gravity sector of the CGHS model, as can be easily seen after a conformal transform $(x^\pm)^k \to x^\pm$. We can either get this expression from Eq.(16) or from Eq.(18) taking $q \to 0$ limit after replacing y_0 with $y_0 + 1/(qk)$.

The black hole solutions are recovered if $M > 0$ and $f_0 = 0$, the case in Fig.1b. In this case, we have $h_+ = 0$ and $h_- = M/(1+q)$ and the scalar field f vanishes identically. The existence of the apparent horizon, which is the space-time point that satisfies $\partial_+ \Omega = 0$ and, furthermore, is the same as the global event horizon in static analysis, is clear from Fig.1b. The indicated physical region to define the inverse of F_{h_-,h_+} includes a point $\frac{d\Omega}{dy} = 0$. The relation between Ω and $y = \ln x$ is

11

calculated to be

$$\int \frac{d\Omega}{k\Omega^{1+q} - M/(1+q)} = \int dy \qquad (20)$$

from Eq.(17), which reduces to

$$\frac{2}{k}(r + \frac{2M}{k} \ln |\frac{k}{2M}r - 1|) = y - y_0$$

in 4-d Einstein gravity (where $q = -1/2$ and $\Omega = r^2$) and

$$\Omega = e^{k(y-y_0)} + \frac{M}{k}$$

in the CGHS model (where $q = 0$), reproducing well-known results. (This result should be carefully compared with the literature due to our somewhat unconventional signature choice.)
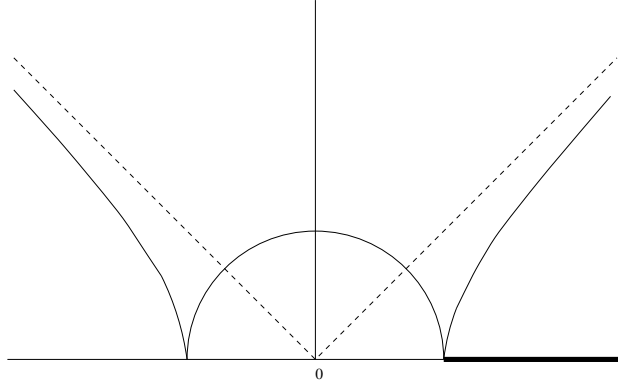

Fig.1c. The plot of F_{h_-,h_+} as a function of $\frac{d\Omega}{dy}$ for $h_\pm = f_0/\sqrt{2(1+q)}$. Dotted lines denote a vacuum space-time. The bold line depicts the region to take the inverse of F_{h_-,h_+}.
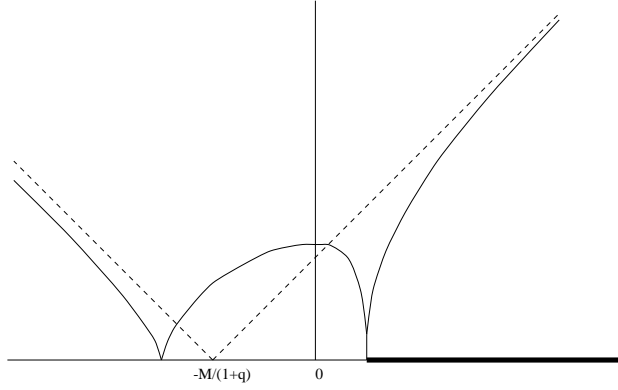

12

Fig.1d. The plot of F_{h_-,h_+} as a function of $\frac{d\Omega}{dy}$ for a typical $h_- > h_+ > 0$ case. Dotted lines denote a black hole geometry with mass M. The bold line shows the region to take the inverse of F_{h_-,h_+}.

In fig.1c, we have $M = 0$ and $f_0 > 0$, resulting $h_\pm = f_0/\sqrt{2(1+q)}$. The solutions in this case are given by

$$\int \frac{d\Omega}{\sqrt{f_0^2/(2+2q) + k^2\Omega^{2+2q}}} = \int dy \tag{21}$$

for Ω and

$$f = \frac{1}{\sqrt{2(1+q)}} \ln\left(\frac{\sqrt{2(1+q)k^2\Omega^{2+2q} + f_0^2} - f_0}{\sqrt{2(1+q)k^2\Omega^{2+2q} + f_0^2} + f_0}\right) + f_1 \tag{22}$$

$$\rho = \frac{1}{2}(1+q)\ln\Omega - \frac{1}{2}y + \rho_0 \tag{23}$$

for f and ρ, respectively. The integral in Eq.(21) can straightforwardly be represented in terms of incomplete beta function or hypergeometric function. Examining the integral for large Ω, we find the asymptotic space-time is the flat vacuum in each theory. Apart from the relation between Ω and $y = \ln x$, Eq.(23) is identical to that in vacuum solutions in each theory. Thus, the metric $g_{\alpha\beta}$ is flat in 2-dimensional sense and all the information regarding the the curvature of space-time is contained in Eq.(21). From Eq.(22), we see that f diverges logarithmically near $\Omega = 0$ and, for large Ω, asymptotically falls off like $1/\Omega^{1+q}$. This asymptotic behavior, in d-dimensional Einstein gravity, translates to fall-off like $1/r^{d-3}$ in terms of geometric radius r. In between these two limits, f is a monotonically increasing function of Ω. The asymptotic solutions for large Ω can also be reproduced directly by taking $G \to 0$ limit, where G is the gravitational constant. This limit is the same as the weak field approximation taking a vacuum metric in each theory as a fixed background metric. The wave equation for f in s-wave sector under this fixed background geometry has general static solutions of the form $f = constant/\Omega^{1+q} + f_1$, a solution describing a point scalar charge sitting at the origin. Thus, Eqs.(21)-(23) are the relativistic generalization of the point scalar charge solution. The strong self-energy of the scalar field near the origin backreacts to the space-time geometry and, in turn, this change softens the power-like divergence of f near the origin into milder logarithmic singularity. We also note that the total d-dimensional gravity-matter action in d-dimensional Einstein gravity vanishes for these solutions . The physical region depicted in Fig.1c does not include the point $\frac{d\Omega}{dy} = 0$, a position where the

13

horizon could have been formed. This lack of trapped region implies the singularity at $\Omega = 0$ is naked. Obviously, black holes are not present in these solutions.

In Fig.1d, we have generic cases of $M > 0$ and $f_0 > 0$, resulting $h_- > h_+ > 0$. The field f diverges logarithmically near $\Omega = 0$ and the space-time at large Ω is asymptotically flat. Actually, F_{h_-,h_+} quite rapidly approaches the F_{h_-,h_+} for a black hole with mass M as $\frac{d\Omega}{dy}$ gets large. (See Appendix for the explicit form of f in this case.) The space-time geometry is also rather similar to the case of point scalar charges, Fig.1c; there are two values of $\frac{d\Omega}{dy}$, for which F_{h_-,h_+} vanishes. The horizon $\frac{d\Omega}{dy} = 0$ is not included in the physical region to take the inverse of F_{h_-,h_+}, leading inevitably into naked singularities. As a result, black holes are not present in these solutions, although M does not vanish. For a better physical understanding of this situation, let us consider the case when $M >> f_0 > 0$, i.e., $h_- \simeq M/(1 + q)$ and h_+, a very small positive number proportional to f_0. Then, we might try to solve the wave equation for f in fixed black hole geometry with mass M, neglecting the backreaction of the self-energy of f on space-time geometry. It is well known that the static solutions for f in this weak field approximation in curved space-time diverge logarithmically at the black hole horizon, $\frac{d\Omega}{dy} = 0$ in Fig.1d (for example, $f \simeq \ln|r - 2M|$ in 4-d Einstein gravity near the horizon $r = 2M$). Since the gravitational self-energy of f correspondingly diverges near the black hole horizon, the weak field approximation is not valid there. What our calculation shows instead is this large self-energy near the horizon backreacts to space-time to cut off the black hole from our view and produces a naked singularity in front of the potential black hole horizon. Indeed for small but finite f_0, the width of the cusp near $\frac{d\Omega}{dy} = h_+$ in Fig.1d gets very small but finite. Only for $f_0 = 0$, the cusp disappears and we recover the black hole geometry. In other words, when black holes try to carry some scalar charge, they end up shutting themselves off from our view, leaving a naked singularity.

To summarize, other than some cosmological solutions with compact range of the geometric radius, all solutions with asymptotically flat space-time have naked singularities, except for black hole solutions (and vacuum solutions) with identically vanishing scalar field f. Thus, we completed the proof of no-hair theorem for all model theories in our consideration. Additionally we showed there are relativistic analog of point scalar charges in each theory.

3. Some Dynamic Solutions

The static point scalar charge solutions can, in some cases, be simply extended to scattering solutions. The s-wave sector of 4-d general relativity is such an example, as shown in this section. The solutions constructed in this fashion describe the situations where either black holes are dynamically formed as a result of the gravitational collapse or the incoming stress-energy flux gets reflected toward the future (null) infinity.

3.1. The Case of 4-d Einstein Gravity in s-wave Sector

The spherically symmetric reduction of 4-d Einstein gravity coupled with a massless scalar field f is the case when $\gamma = 2$, $\mu = -2$, $\lambda = 1$ and $\delta = 1^*$ and, as our action shows, the gravitational constant G satisfies $16\pi G = 1$. Then, the static solutions for $M = 0$, i.e. point scalar charge solutions, are calculated to be

$$\Omega = \frac{e^{-2\rho_0}}{4}(e^{4\rho_0}(\ln(x/x_0))^2 - 4f_0^2), \tag{24}$$

$$\rho = \frac{1}{4}\ln\Omega - \frac{1}{2}\ln x + \rho_0,$$

$$f = \ln(\frac{\sqrt{e^{2\rho_0}\Omega + f_0^2} - f_0}{\sqrt{e^{2\rho_0}\Omega + f_0^2} + f_0}) + f_1 = \ln(\frac{e^{2\rho_0}\ln(x/x_0) - 2f_0}{e^{2\rho_0}\ln(x/x_0) + 2f_0}) + f_1,$$

where ρ_0, f_1 and x_0 are arbitrary constants. For simplicity, we take $x_0 = 1$ and $\rho_0 = 0$ for further discussions. Then, as $\Omega \to \infty$, the behavior of f asymptotically approaches to $-2f_0/r + f_1$ where the geometric radius r is defined to be $\sqrt{\Omega}$. In this limit, we find $r \to (\ln x^+ + \ln x^-)/2$ and the 4-d metric becomes $ds^2 \to -dx^+dx^-/(x^+x^-) - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$. After a conformal transformation $\ln x^\pm \to x^\pm$, we find that the asymptotic space-time is a flat Minkowskian. We note that under weak field approximation taking a Minkowskian metric as a fixed background metric, s-wave sector of static solutions for f is given by $f = -2f_0/r + constant$. The physical property of this solution becomes more transparent if we use r coordinate to describe the geometry of longitudinal space-time

---

*We note here that if we couple a massless fermion field and describe the fermion s-wave sector via a bosonization procedure, the only difference from the above specification of parameters is the value of $\delta = 0$.

instead of conformal coordinates. Then, from Eq.(24), we find that the metric is given by

$$ds^2 = dt^2 - \frac{1}{1 + f_0^2/r^2} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{25}$$

The above-mentioned asymptotic behavior is clear from this equation.

The $M = 0$ case solutions are of particular importance as we can generalize the static solutions to dynamic ones. We note the structure of s-wave sector of 4-d Einstein gravity is very simple as far as the limit $G \to 0$ is concerned. Then, we have a fixed Minkowski space-time and the linear wave equation for f has the general dynamic solution of the form $f = (f_+(x^+) + f_-(x^-))/r$. Comparing it to the static solutions in the same limit, we find that replacing the constant f_0 in static solutions with an arbitrary chiral field generates the general dynamic solutions. This consideration suggests that the scalar charge f_0, even in general relativistic cases, may be a chiral field instead of being a strict constant, similar to what happens in $G \to 0$ limit. In the framework of general relativity, however, this simple extension is not possible in general, since there can be a non-trivial corrections to Eqs.(24) of the order of $\partial_\pm f_0$. In case of asymptotically steady incoming and outgoing stress-energy flux, though, the corrections are easily found. Forgetting about global boundary conditions, the result of this extension is

$$\Omega = \frac{1}{4}(\ln x)^2 - \frac{1}{4}(k_+ \ln x^+ - k_- \ln x^- + q_0)^2, \tag{26}$$

$$\rho = \frac{1}{4}\ln\Omega - \frac{1}{2}\ln x + \frac{1}{2}\ln(1 + k_+ k_-),$$

$$f = \ln(\frac{\ln x - (k_+ \ln x^+ - k_- \ln x^- + q_0)}{\ln x + (k_+ \ln x^+ - k_- \ln x^- + q_0)}) + f_1,$$

where $k_\pm$ and q_0 are constants. We can straightforwardly verify that these solutions satisfy field equations derived from Eq.(5) and the corresponding gauge constraints. Apart from the additional constant term in the expression for ρ, the correction term originating from $\partial_\pm f$, Eqs.(26) are the same as Eqs.(24) with $f_0 = (q_0 + k_- \ln x^+ - k_+ \ln x^-)/2$. The correction turns out to be rather simple in this case where the charge f_0 has terms only up to linear terms in $\ln x^\pm$, which are asymptotically flat conformal coordinates near the past or future infinity. The asymptotic stress-energy tensor averaged over the transversal sphere in a conformal coordinate system $v = \ln x^+$ ($u = \ln x^-$) that becomes asymptotically flat near the past (future) infinity is calculated to be $T_{vv} = k_+^2$ ($T_{uu} = k_-^2$). Thus, $k_\pm^2$ are

16

interpreted to be an incoming and an outgoing energy flux, respectively. The constant q_0 represents the background component of the scalar charge. The $q_0 = 0$ case of Eqs.(26) was reported in mathematics literature [9] (named scale invariant solutions) and in physics literature [13] where the solutions were used to investigate the violation of the cosmic censorship hypothesis. For scattering situations where incoming and outgoing flux can coexist, a slightly generalized version (26) proves to be useful. Additionally, the presence of q_0 term enables us to consider the time evolution of the (multiple) square-type incoming energy pulses by successively gluing our solutions, unless a black hole is formed in an intermediate stage. To name a few other applications possible with our result, we can construct various scattering solutions, cosmological solutions and point particle solutions with time-varying charge at the origin, depending on the boundary conditions and initial conditions.

To illustrate a black hole formation in this system, we consider this physical situation; in the asymptotic (null) past, we turn on the constant incoming stress-energy flux, inject it in a spherically symmetric fashion for a time duration and turn it off. After a dynamical evolution, we will be interested in what comes out in the asymptotic (null) future. The physical region of space-time in our consideration is specified by the requirement $\Omega \geq 0$, since the angular coordinates should not have time-like signature. Thus, the natural boundary is the origin, $\Omega = 0$. In the limit $G \to 0$, with a fixed Minkowskian background, the situation is represented by the solutions depicted in Fig.2 and given by

$$
\begin{aligned}
f &= 0 & &\text{I} \\
f &= \frac{-kv}{r} & &\text{II} \\
f &= -2k & &\text{III} \\
f &= \frac{-k(v_0+u)}{r} & &\text{IV} \\
f &= 0 & &\text{V} \\
f &= \frac{-kv_0}{r} & &\text{VI}
\end{aligned}
\tag{27}
$$

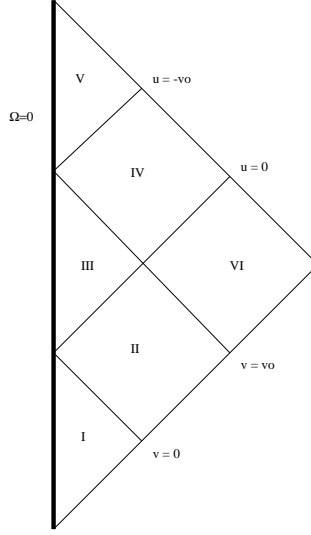where we use flat coordinates $v = \ln x^+$ and $u = \ln x^-$, thereby $\Omega = r^2 = ((u+v)/2)^2$.

Fig.2. The Penrose diagram shows the space-time region in fixed Minkowskian background. The bold line denotes the space-time boundary $\Omega = 0$. The constant incoming stress-energy flux is turned on during the null time $v = 0$ and $v = v_0$.

The initial data on the past null infinity were so chosen to describe the turn-on of the constant incoming stress-energy flux $T_{vv} = k^2$ there at $v = 0$ and the subsequent turn-off at $v = v_0 \geq 0$. We also imposed a boundary condition by requiring the field f to be finite along $\Omega = 0$. This approximate solution in a *fixed* Minkowskian background has a property that any unbounded amounts of the incoming stress-energy flux are totally reflected off from the origin into the future null infinity. However, this picture qualitatively changes as we consider the exact solutions focusing on the space-time geometry change due to the stress-energy of the scalar field. Under the same choice of the initial data on the past null infinity and the boundary condition along $\Omega = 0$, we can construct the following relativistic solutions. For Ω field, we find

$$\Omega = \tfrac{1}{4}(u+v)^2 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{I} \qquad\qquad (28)$$
$$\Omega = \tfrac{1}{4}(u+v)^2 - \tfrac{1}{4}k^2v^2 \qquad\qquad\qquad\qquad\quad \text{II}$$
$$\Omega = \tfrac{1}{4}(\sqrt{1-k^2}v + \tfrac{u}{\sqrt{1-k^2}})^2 \qquad\qquad\qquad \text{III}$$
$$\Omega = \tfrac{1}{4}((1-k^2)v + \tfrac{u}{1-k^2} + k^2v_0)^2 - \tfrac{1}{4}k^2(v_0 + \tfrac{u}{1-k^2})^2 \quad \text{IV}$$
$$\Omega = \tfrac{1}{4}((1-k^2)v + \tfrac{u}{1-k^2} + k^2v_0)^2 \qquad\qquad\quad\; \text{V}$$

and

$$f = 0 \qquad\qquad\qquad\qquad\qquad \text{I} \qquad\qquad (29)$$

18

$$f = \ln\left(\frac{v+u-kv}{v+u+kv}\right) \qquad \text{II}$$

$$f = \ln\left(\frac{1-k}{1+k}\right) \qquad \text{III}$$

$$f = \ln\left(\frac{(1-k^2)v+u/(1-k^2)+k^2v_0-k(v_0+u/(1-k^2))}{(1-k^2)v+u/(1-k^2)+k^2v_0+k(v_0+u/(1-k^2))}\right) \quad \text{IV}$$

$$f = 0 \qquad \text{V}$$

for f field. The function ρ can be read off from Eqs.(26). We note the limit $v_0 \to \infty$, in which we do not turn off the incoming flux, leaves us with the regions I, II and III. Our solutions in this limit were previously discovered and discussed by some authors [9] [12] [5]. The solution in the region VI is rather complicated and, in generic cases, can not be obtained using Eqs.(26) or the conformally transformed version of them for the reasons explained later. However, we can obtain the region IV solution by matching the III-IV boundary and from a reasonable physical assumption that there is no infalling stress-energy flux in the region IV since the asymptotic incoming stress-energy flux is turned-off for $v > v_0$. Eqs.(29) reduce, after taking the limit $G \to 0$ (or $k \ll 1$) and under the leading order approximation, to Eqs.(27) and the geometry is almost Minkowskian, as expected.
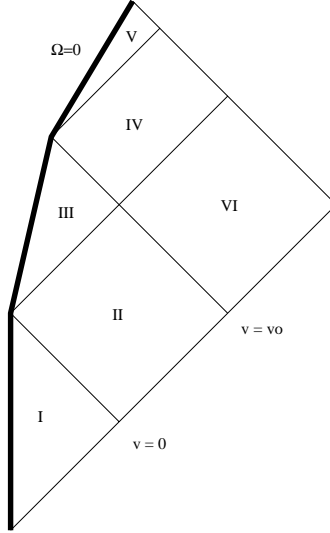

Fig.3. The Penrose diagram for a typical subcritical case. The boundary $\Omega = 0$, the bold line shown above, remains time-like throughout the entire space-time.

The region I, bounded by the past null infinity, $v = 0$ and the origin $u = -v$, represents the Minkowski space before the turn-on of the constant incoming flux. Regardless of v_0, the qualitative properties of our solutions beyond the region I are distinctively different for $k < 1$ and $k > 1$ (notice $T_{vv} = k^2$).

19

For $k < 1$, depicted in Fig.3, all the regions I-VI exist. The region II, bounded by $v = v_0$, $v = 0$ and the past null infinity, represents the propagation of the incoming particles before any of them hits the boundary $\Omega = 0$. The region III represents the region where incoming particles and outgoing particles reflected off from $\Omega = 0$ coexist. The net energy flux cancels as a result, and we have a flat space-time throughout the region and f field, a constant. However, with respect to the flat Minkowskian in the region I, the flat space-time in the region III is Lorentz boosted with relative speed $v/c = k^2/(2 - k^2)$. The region IV contains the outgoing particles further propagating toward the future null infinity. In the asymptotic out region, which includes the asymptotically flat future null infinity in the region IV and the strictly flat region V, there is additional Lorentz boost with respect to the flat region III with relative speed $v/c = k^2/(2 - k^2)$ and a translation $k^2 v_0$. Due to these Lorentz boosts, the original energy pulse looks time-contracted and amplified after the reflection off the origin from the point of view of asymptotic in-observers. On the other hand, from the point of view of asymptotic out-observers who use $U(u) = u/(1 - k^2)$ and $V(v) = (1 - k^2)v$, the outgoing stress-energy flux is $T_{UU} = k^2$ and the duration of the pulse is v_0, exactly the same as what an asymptotic in-observer sent in. Thus, the total outgoing energy carried by the outgoing particles in the region IV is the same as the total incoming energy thrown in initially. From the conservation of energy, we can infer that no further out-going energy flux comes out through the future null infinity in the region VI. We also note that the amount of the translation of coordinates from the region I to the regions IV and V, $k^2 v_0$, is proportional to the total injected energy. In summary, for $k < 1$, the path of the origin remains time-like throughout the whole space-time and incoming particles are scattered forward into the future null infinity, just as in the subcritical regime of the numerical simulations.
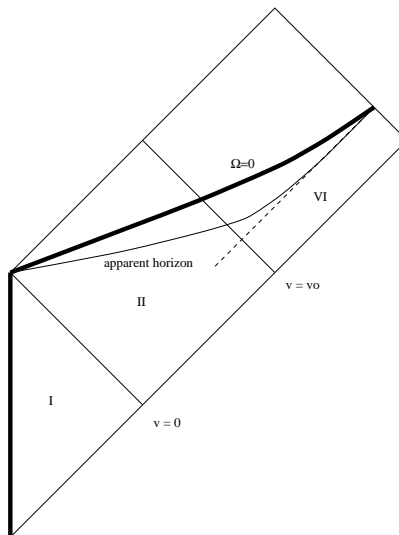
Fig.4. The Penrose diagram for a typical supercritical geometry. The boundary $\Omega = 0$, the bold line, becomes space-like for $v > 0$ and the apparent horizon forms. The details shown in the region VI are conjectural.

If $k > 1$, as shown in Fig.4, the path of the origin becomes space-like and form a trapped surface in the region II. In this case, the regions III, IV and V disappear and our solutions in the region II become exactly the same as the one obtained in [9]. As explained in detail in [9], the resulting space-time is a dynamic black hole with increasing mass, for we do not turn off the constant incoming flux up until $v = v_0$. Thus, this corresponds to the supercritical phase of this scattering system. At $k = 1$, which can thus be interpreted as a black hole formation threshold, or a phase transition point, the path of the origin becomes light-like in the region II. As the numerical studies and the above considerations suggest, the order parameter of this system is the black hole mass, which vanishes for $k < 1$ and becomes non-vanishing for $k > 1$. Then, the important physical quantity to compute, given our exact solutions in supercritical regime, is the critical exponent. The geometric radius $r = r_A(v)$ of the apparent horizon of the dynamic black hole in supercritical case, determined by $\partial_v r = 0$, is calculated to be

$$r_A(v) = \frac{1}{2}(k-1)^{1/2}(k+1)^{1/2}kv. \tag{30}$$

The $1/2$ times the value of this corresponds to the apparent mass M_A of the dynamic black hole. The linear dependence on v is understandable as it is the time duration between the turn-on of the incoming flux and the reference time v. We also note

that the angular integrated incoming energy flux is $4\pi k^2 = \frac{1}{4}k^2/G$. Defining the transition point $p^* = 1$ and $p = k$, we find that the critical exponent in this case is $\Delta = 1/2$ in a scaling relation $M_A \simeq (p - p^*)^\Delta$.

In the numerical study [1], the scaling relation is given for the asymptotically static black hole mass M_{BH}, measured in the future (null) infinity, and p. In our case, on the other hand, since we do not know the solutions in the region VI, the scaling relation between M_A and M_{BH} is not available. However, it is possible to show that, in generic cases, we can not choose static solutions in the region VI (in the case of $G \to 0$ limit, the solutions in the region VI are indeed static as shown in Eqs.(27)), thereby identifying M_{BH} and M_A. To show this, we recall the general static solutions in the region VI are given by

$$U(u) + V(v) = 2 \int \frac{r dr}{F_{h_-,h_+}^{-1}(r)} \tag{31}$$

from section 2 in conformal gauge. Along II-VI boundary, $v = v_0$, we have to satisfy

$$\Omega = \frac{1}{4}(v_0 + u)^2 - \frac{1}{4}k^2 v_0^2 = (r(u))^2 \tag{32}$$

and

$$\partial_v \Omega = \frac{1}{2}u + \frac{1}{2}(1 - k^2)v_0. \tag{33}$$

By requiring v coordinate to be C^∞ along the past null infinity, we find $V(v) = v$. Eq.(32) determines the function $U(u)$ via

$$U(u) + v_0 = 2 \int^{r=r(u)} \frac{r dr}{F_{h_-,h_+}^{-1}(r)} \tag{34}$$

where the function $r(u)$ is given in Eq.(32). After some calculations, Eq.(33) becomes

$$F_{h_-,h_+}(r) = \sqrt{(r - \frac{1}{2}k^2 v_0)^2 - \frac{1}{4}k^2 v_0^2} \tag{35}$$

and this should determine $h_\pm$. Unless the scalar charge kv_0 goes to zero, Eq.(35) can not be satisfied for any values of $h_\pm$. Thus, we have shown the above statement for generic cases. The only exceptional cases are when the scalar charge vanishes after $v = v_0$. A notable example of such exceptions is the shock wave injections [10]. In this case, we take $v_0 \to 0$ limit for fixed $k^2 v_0$, to make the injected energy finite. Then, we see that the region II disappears and kv_0 vanishes as $v_0 \to 0$. The exact solutions, obtained from the above consideration and shown in Fig.5, are given by

$$\Omega = \frac{1}{4}(u + v)^2 \tag{36}$$

for the region I, a flat Minkowskian, and

$$r + 2m \ln|\frac{r}{2m} - 1| = \frac{1}{2}(U(u) + v)$$

along with $U(u) = u + 4m \ln|u/4m - 1|$ and $m = k^2 v_0/4$ in the region VI, the Schwarzschild geometry. The scalar field vanishes in the whole region except the line $v = 0$ (shock-wave). Recalling Eq.(30) and the divergence of k, this is an example of extremely supercritical cases and we indeed have $M_A = M_{BH}$.
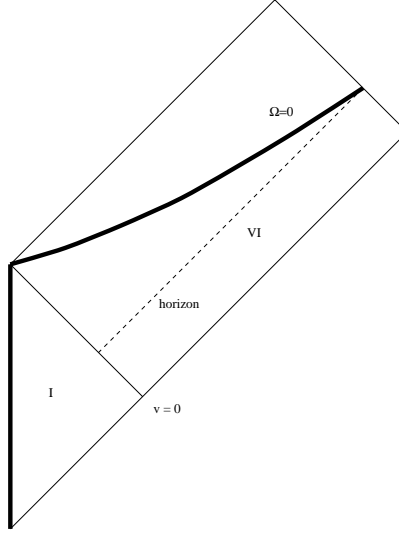

Fig.5. The Penrose diagram for a shock wave injection along $v = 0$. The geometry for $v > v_0$ is described by Schwarzschild metric and the geometry below that is flat Minkowskian.

In supercritical cases, we consequently infer that the space-time geometry in the region VI goes through some transient non-static period as long as the scalar charge does not vanish for $v > v_0$. The no-hair theorem supports this result. For a given finite incoming energy, the theorem insists the only possible candidates for the asymptotic final state geometry are black holes with no scalar charge. Thus the role of the transient period is to bleach the static component of the residual scalar charge. During this process, the apparent horizon that was initially space-like at the turn-off time will settle down to a future null direction, the asymptotic final horizon, slightly changing its geometric radius. The numerically obtained $M_{BH} \simeq (p - p^*)^\Delta$ with $\Delta \simeq 0.37$ is very difficult to calculate, as the scaling relation between M_{BH} and M_A gets complicated through this process. In the subcritical cases, we showed that

23

the energy conservation prohibited the emission of out-going flux in the region VI. In contrast, the numerically established scaling behavior for the asymptotic black hole mass M_{BH} implies that some energy flux escapes into the future null infinity in the region VI for supercritical (especially near critical) cases.

3.2. The Case of Generic Gravity Theories

To apply our method of the extension to other gravity theories, we have to get a dynamic solution in $G \to 0$ limit, taking a vacuum metric as a fixed background. For example, in d-dimensional Einstein gravity, the linear wave equation in (t, r) coordinates in the limit is given by

$$\partial_t^2 f^* - \partial_r^2 f^* + \frac{1}{2}(d-2)(d-4)\frac{1}{r^2}f^* = 0 \tag{37}$$

where $f = f^*/r^{(d-2)/2}$ and r is the geometric radius. The vacuum solution for the space-time satisfies $\Omega = r^{d-2}$. We immediately find that, unless $d = 4$, the solutions are quite complicated. Additionally, the static solutions in each case are $f = f_0/r^{d-3} + f_1$ where f_0 and f_1 are constants. Thus, one can imagine setting $f_0 = r^{(d-4)/2}f^*$ in our static solutions for f^* that produces the constant incoming (outgoing) flux in the past (future) null infinity, for the relativistic extension. Whether this procedure works or not is not yet clear.

In case of the CGHS model for pure gravity sector, or $d \to \infty$ limit, the linear wave equation is

$$\partial_t^2 f^* - \partial_x^2 f^* + \frac{k^2}{4}f^* = 0 \tag{38}$$

where $f = \exp(-kx/2)f^*$. Here the geometric background is the linear dilaton vacuum $\Omega = \exp(kx)$ and x is the asymptotically flat spatial coordinate. The major difficulty in this case is the existence of mass term, causing f^* to be non-chiral even for the asymptotic infinities.

One interesting point for theories with $d > 4$ (including $d = \infty$ case) is that $\sqrt{\Omega}f$ vanishes as $\Omega \to \infty$ for non-trivial (non-constant) static solutions f. This implies the static scalar charge component of solutions completely decouples from the dynamic solutions; the information about the static component can not be contained in the initial data specified along the past null infinities, since it vanishes too fast. The $d = \infty$ case shows this most clearly. The physical frequency spectrum of f^* that is massive requires the wave length becomes purely imaginary number if we set the

frequency to be zero. In fact, the physically allowed frequency is larger than the mass in this theory. Thus, the transient behavior of static scalar charge bleaching in 4-d Einstein gravity after the turn-off of the incoming energy flux is expected to be absent in theories with $d > 4$.

4. Discussions

Our derivation of the general static solutions in section 2 relies heavily on the existence of three rigid symmetries of the action. Among these symmetries, there is a static remnant from the underlying conformal invariance, which can thus be made local. However, the symmetry whose charge relates to a black hole mass does not share this property. In the CGHS model with $\delta = 0$, we have additional rigid symmetry $\Omega \to \Omega + A$ where A is a constant. Additionally, this symmetry and $f \to f + B$ can both be made local. These local symmetries resulting from the underlying 2-d Poincare current algebraic symmetry are enough to determine the general solutions of the CGHS model. One natural question, then, for other gravity theories is whether it is possible to find three rigid symmetries that can be gauged as what happens in CGHS model. The existence of them will be helpful in finding the general solutions in each theory.

The dynamic solutions in s-wave 4-d Einstein gravity obtained in section 3 are interesting from many point of view. First, the dynamics of the point $\Omega = 0$ shows a remarkably similar behavior to that of dynamical moving mirror considered in [14] for the CGHS model. Trajectories of both points dynamically reacts to the incoming energy flux. Furthermore, depending on the asymptotic incoming energy flux, both show the subcritical behavior where the trajectory is strictly time-like and the supercritical behavior where the trajectory becomes space-like. Thus, in some sense, the seemingly extraneous introduction of reflecting dynamic boundary (moving mirror) in the CGHS model makes the model behave more like phenomenologically interesting s-wave sector of 4-d Einstein gravity. In the latter theory, the dynamics of $\Omega = 0$ is a necessary consequence of the theory, as we showed. Second, it will be very interesting to study the quantum field theory on a near critical but subcritical geometry. In this case, the incoming and outgoing null coordinates are related by very large Lorentz boost, somewhat similar to the infinite red shift near the black

hole horizon. Unlike the black hole case, however, no information should be lost in the case of subcritical solutions and, additionally, the mode expansion for the free scalar field on the asymptotic (null) infinities are well defined. Therefore, the near critical and subcritical solutions provide a nice background geometry to study gravitational interactions between incoming and outgoing quantum fluctuations.

Our third interest lies in the phase transition itself. The black hole phase transition in the CGHS model is not as difficult as in the spherically symmetric 4-d Einstein gravity. The complicated transient behavior is absent in this case, due to the simplified dynamics of the model. Therefore, it would be possible to directly glue a static dilatonic black hole at $v = v_0$ and thereby getting a relation $M_A \simeq M_{BH}$ if we had considered the CGHS model from the the outset. An interesting observation in this regard is the critical exponent 0.5 for the scaling relation for the apparent mass in s-wave 4-d Einstein gravity is the same as the critical exponent obtained by Strominger and Thorlacius in the CGHS model [6]. There is a reason for this connection as suggested in [11]. The pure gravity sector of the CGHS model, other than being a target space effective action from string theory, can be considered as a leading order theory in the $1/d$-expansion of the spherically symmetric d-dimensional Einstein gravity. We can, therefore, adopt $1/d$-expansion and consider the leading order behavior in the description of the complex transient process in 4-d Einstein gravity. As a zeroth order approximation, we glue a CGHS black hole directly to our solutions in the region II to deduce the approximate scaling relation $M_A \simeq M_{BH}$. (See section 3.2.) Thus, the leading order approximation of the exact critical exponent for square pulse-type incoming energy flux is now calculated to be 0.5. Since the next order correction to the critical exponent is expected to be an order of $1/d = 0.25$ and the numerically calculated value is about 0.37, our leading order value, 0.5, seems plausible. Furthermore, it is conceivable that the exact critical exponent ($M_{BH} \simeq |p - p^*|^\Delta$) for pulse-type incoming energy flux in d-dimensional spherically symmetric Einstein gravity for $d > 4$ would be 0.5, considering the presumed lack of transient behavior discussed in section 3.2. In many other cases of phase transitions in condensed matter physics, the critical exponent gets the scaling violation for only lower dimensional cases. It will be an interesting exercise to verify this conjecture and, additionally, develop a systematic perturbation theory with a dimensionless expansion parameter $1/d$ to tackle other difficult problems in 4-dimensional gravity.

Acknowledgement

Appendix. The Alternative Derivation of Static Solutions

Instead of choosing a conformal gauge, we can choose a gauge used in [5] where the general solutions of pure gravity sector in our consideration were obtained to prove Birkhoff's theorem. From now on, we follow the convention of the reference. The metric tensor in this gauge is given by

$$g_{\alpha\beta} = \begin{bmatrix} -\alpha^2 & 0 \\ 0 & \beta^2 \end{bmatrix}. \tag{39}$$

Furthermore, we choose coordinates in such a way that $x^1 = \Omega = \exp(-2\phi)$ and require $[\partial_0, \partial_1] = 0$. We note that the signature convention, the time component being negative, is different from section 2 to closely follow the reference [5].

The resulting equations of motion can be written as

$$\frac{\partial_1\alpha}{\alpha} + \frac{\gamma}{8\Omega} - \frac{\beta^2}{2}\Omega V(\Omega) - \frac{\Omega^\delta}{4}[(\partial_1 f)^2 + \frac{\beta^2}{\alpha^2}(\partial_0 f)^2] = 0 \tag{40}$$

$$\frac{\partial_1\beta}{\beta} + \frac{\gamma}{8\Omega} + \frac{\beta^2}{2}\Omega V(\Omega) - \frac{\Omega^\delta}{4}[(\partial_1 f)^2 + \frac{\beta^2}{\alpha^2}(\partial_0 f)^2] = 0 \tag{41}$$

$$\partial_1(\frac{\alpha}{\beta}\Omega^\delta\partial_1 f) - \partial_0(\frac{\beta}{\alpha}\Omega^\delta\partial_0 f) = 0 \tag{42}$$

along with a gauge constraint

$$\frac{\partial_0\beta}{\beta} - \frac{1}{2}\Omega^\delta\partial_0 f\partial_1 f = 0. \tag{43}$$

Since we are interested in getting general *static* solutions, we require $\partial_0 f = 0$ and $\partial_0\beta = 0$. Thus the gauge constraint (43) is automatically satisfied. Furthermore, Eq.(42) can be directly integrated to yield

$$\Omega^\delta\partial_1 f = f_0\frac{\beta}{\alpha}. \tag{44}$$

Adding (40) and (41) gives

$$\partial_1 \ln A + \frac{\gamma}{4\Omega} - \frac{f_0^2}{2\Omega}\frac{1}{B^2} = 0 \tag{45}$$

while subtracting them produces

$$\partial_1 \ln B = \mu \Omega^{1-\lambda} \frac{A}{B} \tag{46}$$

where we define $A = \alpha\beta$ and $B = \alpha/\beta$. We also set $\delta = 1$ and $V(\Omega) = \mu\Omega^{-\lambda}$. Introducing $\bar{A}$ via $A = \Omega^{\lambda-2}\bar{A}$, we can rewrite above equations as follows.

$$\frac{d}{d\phi}B = -2\mu\bar{A} \tag{47}$$

$$\frac{1}{\bar{A}}\frac{d}{d\phi}\bar{A} = -(2 + 2q + \frac{f_0^2}{B^2}) \tag{48}$$

Here, as in section 2, q is defined as $q = 1 - \lambda - \gamma/4$. We can plug Eq.(47) into Eq.(48) and straightforwardly integrate it once to obtain

$$-\frac{d}{d\phi}B = \frac{1}{B}(2(1+q)B^2 + 2MB - f_0^2) \tag{49}$$

where M is the constant of integration. After integrating this once more, we finally get

$$k\Omega^{1+q} = |h_+ - B|^{\frac{h_+}{h_+ + h_-}}|h_- + B|^{\frac{h_-}{h_+ + h_-}} = F_{h_-,h_+}(B) \tag{50}$$

where k is a constant of integration and

$$h_\pm = \frac{1}{2(1+q)}(\sqrt{M^2 + 2(1+q)f_0^2} \mp M).$$

We note that Eq.(50) is exactly the same as Eq.(16) from section 2 if B is replaced by $\frac{d\Omega}{dy}$. Using Eq.(47), A can be determined as

$$A = \frac{\Omega^{\lambda-1}}{\mu}\frac{dB}{d\Omega}. \tag{51}$$

Eqs. (50) and (51) are our main results. They implicitly solve the metric in terms of Ω. Then f can be directly integrated using Eq.(44).

Choosing B as a spatial coordinate in favor of Ω gives more explicit solutions. Then, the metric is computed to yield

$$ds^2 = -\frac{B\Omega^{\lambda-1}}{\mu}(\frac{d\Omega}{dB})^{-1}dt^2 + \frac{\Omega^{\lambda-1}}{\mu B}(\frac{d\Omega}{dB})dB^2 \tag{52}$$

28

where the function Ω in terms of B is given in Eq.(50). Using Eqs.(44) and (50), the scalar field f can be explicitly solved as

$$f = \frac{f_0}{\sqrt{M^2 + 2(1+q)f_0^2}} \ln|\frac{B - h_+}{B + h_-}| + f_1.$$ (53)

Setting $B = \frac{d\Omega}{dy}$ and $M = 0$ in the above equation reproduces Eq.(22) in section 2. The solutions of [7] are recovered if we set $1 + q = 1/2$ and $\lambda = 1$, the case of 4-d Einstein gravity, as long as $M \neq 0$. We see that it is again possible to derive the *general* static solutions.

References

[1] M.W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).

[2] C.G. Callan, S.B. Giddings, J.A. Harvey and A. Strominger, Phys. Rev. **D45**, R1005 (1992).

[3] S. Giddings, *"Quantum Mechanics of Black Holes,"* lectures at the Trieste Summer School on High Energy Physics, July 1994, hep-th-9412138. A. Strominger, *"Les Houches Lectures on Black Holes."* presented at the 1994 Les Houches Summer School, hep-th-9501071.

[4] D. Christodoulou, Comm. Math. Phys. **105**, 337 (1986);**106**, 587 (1986); **109**, 613 (1987).

[5] Y. Kiem, Princeton Preprint, PUPT-1484, hep-th-9407100.

[6] A. Strominger and L. Thorlacius, Phys. Rev. Lett. **72**, 1584 (1994).

[7] A.I. Janis, D.C. Robinson and J. Winicour, Phys. Rev. **186**, 1729 (1969); J.D. Bekenstein, Ann. Phys. (N.Y.) **82**, 535 (1974).

[8] A.B. Abrahams and C.R. Evans, Phys. Rev. Lett. **70**, 2980 (1993).

[9] D. Christodoulou, Comm. Pure Appl. Math. **46**, 1131 (1993).

[10] G. t'Hooft and T. Dray, Comm. Math. Phys. **99**, 613 (1985).

[11] Y. Kiem, Phys. Lett. **B322** , 323 (1994).

[12] P. Brady, Class. Quant. Grav. **11**, 1125 (1994).

[13] M.D. Roberts, Gen. Rel. and Grav. **21**, 907 (1989).

[14] T.D. Chung and H. Verlinde, Nucl. Phys. **B418**, 305 (1994).